Derivation of the 5D Einstein-Hilbert Action and Its Reduction in Kaluza-Klein Theory

Douglas G. Stevenson
stevensonfluxinformationtheory.com

March 2026

Contents

1 Introduction

The Einstein-Hilbert action is the starting point for general relativity. In Kaluza-Klein theory, we begin with the 5D version of this action and perform dimensional reduction over a compactified extra dimension to obtain an effective 4D theory containing both gravity and electromagnetism.

This document derives the 5D Einstein-Hilbert action and shows its explicit reduction.

2 The 5D Einstein-Hilbert Action

The 5D Einstein-Hilbert action is given by

$$S_5 = \frac{1}{16\pi G_5} \int d^5x \sqrt{-G} R^{(5)}, \tag{1}$$

where:

- G_{AB} is the 5D metric ($A, B = 0, 1, 2, 3, 5$),
- $-G = \det(G_{AB})$,
- $R^{(5)}$ is the 5D Ricci scalar,
- G_5 is the 5D gravitational constant.

The factor $\frac{1}{16\pi G_5}$ ensures the correct normalization in five dimensions [1].

3 Kaluza-Klein Metric Ansatz

We adopt the standard Kaluza-Klein ansatz (cylinder condition: the metric is independent of the extra coordinate x^5):

$$ds_5^2 = g_{\mu\nu}(x)dx^\mu dx^\nu + \phi^2(x)(dx^5 + A_\mu(x)dx^\mu)^2, \tag{2}$$

where:

- $g_{\mu\nu}(x)$ is the 4D metric,
- $A_\mu(x)$ is the electromagnetic vector potential,
- $\phi(x)$ is the dilaton (radion) field.

The extra dimension x^5 is compactified on a circle of radius R_c with periodicity $x^5 \sim x^5 + 2\pi R_c$ [5].

3.1 Determinant and Volume Element

The 5D metric determinant satisfies $\sqrt{-G} = \phi\sqrt{-g}$, where $g = \det(g_{\mu\nu})$ is the 4D metric determinant.

3.2 Reduction of the 5D Ricci Scalar

The 5D Ricci scalar $R^{(5)}$ can be expressed in terms of 4D quantities plus electromagnetic and scalar contributions [4]. After substitution of the metric ansatz and integration over the compact coordinate x^5 (length $2\pi R_c$), the action reduces to

$$S_4 = \int d^4x\sqrt{-g}\left[\frac{\phi}{16\pi G_4}R^{(4)} - \frac{\phi^3}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\phi}\partial_\mu\phi\partial^\mu\phi\right], \tag{3}$$

where $G_4 = G_5/(2\pi R_c)$ is the effective 4D Newton constant, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength.

3.3 Resulting 4D Theory

The reduced action contains three parts:

- The Einstein-Hilbert term for gravity (scaled by ϕ),
- The Maxwell term for electromagnetism (scaled by ϕ^3),
- A kinetic term for the dilaton field ϕ.

In the simplest case where the dilaton is constant ($\phi = 1$), the action reduces to the standard Einstein-Maxwell action:

$$S_4 = \int d^4x\sqrt{-g}\left[\frac{1}{16\pi G_4}R^{(4)} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right]. \tag{4}$$

Thus, both gravity and electromagnetism emerge from pure 5D geometry.

4 Connection to SFIT

Kaluza-Klein achieves unification through a compactified extra dimension and geometric reduction of the 5D Einstein-Hilbert action. SFIT achieves unification through a dynamic information-carrying flux in four dimensions, introducing a non-reciprocal, time-dependent metric correction at frequency ν_{res}.

While Kaluza-Klein is purely geometric and operates at the Planck scale, SFIT is dynamical and information-theoretic, with clear laboratory-scale predictions (1.20134 mHz resonance, testable in ultra-cold neutron experiments).

A possible synthesis is that Kaluza-Klein describes the ultraviolet geometric unification, while SFIT describes the effective low-energy resonant behavior when the higher-dimensional structure interacts with a macroscopic gravitational field [2].

5 Conclusion

The 5D Einstein-Hilbert action, when reduced under the Kaluza-Klein ansatz, yields an effective 4D theory containing Einstein gravity, Maxwell electromagnetism, and a scalar field [3]. This derivation demonstrates that gravity and electromagnetism can emerge from a single geometric theory in five dimensions.

This completes the classical geometric unification originally envisioned by Kaluza and Klein. SFIT offers a complementary modern approach based on information dynamics at laboratory scales.

References

[1] D. Saha and A. K. Sanyal, "Vulnerability of $f(Q)$ gravity theory and a possible resolution," *Academia Quantum*, vol. 2, no. 1, 2025, doi: https://doi.org/10.20935/AcadQuant7594.

[2] S. Kornowski, "The correct interpretation of the Kaluza-Klein theory," *viXra*, 2014, url: https://vixra.org/pdf/1412.0008v4.pdf.

[3] T. Appelquist, A. Chodos, and P. Freund, *Modern Kaluza-Klein Theories*, Addison-Wesley, 1987.

[4] P. Wang, "Kaluza-Klein dimensional reduction and Gauss-Codazzi-Ricci equations," *International Journal of Modern Physics A*, vol. 24, no. 06, pp. 1207–1220, 2009, doi: https://doi.org/10.1142/s0217751x09042967.

[5] O. Klein, "Quantentheorie und fünfdimensionale Relativitätstheorie," *Surveys in High Energy Physics*, vol. 5, no. 3, pp. 241–244, 1986, doi: https://doi.org/10.1080/01422418608228771.